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Uniform Convergence of Modified Hermite–Fejér Interpolation Processes

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1. INTRODUCTION

1.1. In 1960 P. Turán [5] considered for $f \in C(K)$, $K := [-1, 1]$, every $n \in \mathbb{N}$ and $j(n) \in \{1, ..., n\}$, modified Hermite–Fejér interpolation polynomials $H^*_{j(n)}(f, A)$ of degree $\leqslant 2n-2$ based on the nodes of a triangular matrix A whose nth row is given by

$$-1 \leqslant x_{nn} < \cdots < x_{1n} \leqslant 1 \tag{1.1}$$

(in the sequel we often omit the index n). The polynomials $H^*_{j(n)}(f, A)$ are uniquely determined by the conditions

$$\begin{aligned} &H^*_{j(n)}(f, A)(x_k) = f(x_k) && (k = 1, ..., n), \\ &H^*_{j(n)}(f, A)'(x_k) = 0 && (k = 1, ..., n \text{ and } k \neq j(n)) \end{aligned} \tag{1.2}$$

and can be represented by

$$H^*_{j(n)}(f, A) = H_n(f, A) + R_{j(n)}(f, A), \tag{1.3}$$

where $H_n(f, A)$ is the Hermite–Fejér interpolation polynomial of degree $\leqslant 2n-1$ satisfying

$$\left.\begin{aligned} &H_n(f, A)(x_k) = f(x_k) \\ &H_n(f, A)'(x_k) = 0 \end{aligned}\right\} \quad (k = 1, ..., n),$$

and the remainder $R_{j(n)}(f, A)$ is represented by

$$R_{j(n)}(f, A)(x) \equiv \frac{\omega_n^2(x)}{x - x_{j(n)}} \cdot \sum_{k=1}^{n} \frac{\omega_n''(x_k)}{(\omega_n'(x_k))^3}, \tag{1.4}$$

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where ω_n is given by

$$\omega_n(x) \equiv \prod_{k=1}^{n} (x - x_k).$$

1.2. Using the special matrix T whose nth row is given by

$$x_{kn} = \cos\left(\frac{2k-1}{2n}\pi\right) \qquad (k = 1, ..., n) \tag{1.5}$$

P. Turán proved the unexpected result (see [5, Theorem 1])

$$\lim_{n\to\infty} \|H^*_{j(n)}(f, T) - f\| = 0$$
$$\Leftrightarrow \int_{-1}^{1} \frac{x \cdot f(x)}{\sqrt{1-x^2}}\, dx = 0 \tag{1.6}$$

($\|\cdot\|$ denotes the usual maximum-norm of $C(K)$) for every $f \in C(K)$ and "exceptional-point" sequences $(x_{j(n)})_{n\in\mathbb{N}}$ satisfying an "ε-restriction," that means

$$|x_{j(n)}| \leqslant 1 - \varepsilon \tag{1.7}$$

for suitable $0 < \varepsilon < 1$ and all sufficiently large n.

1.3. In answer to this result V. Kumar and K. K. Mathur [2] in 1980 asserted that for every $f \in C(K)$ and every exceptional-point sequence uniform convergence

$$\lim_{n\to\infty} \|H^*_{j(n)}(f, B_i) - f\| = 0 \tag{1.8}$$

can be achieved by use of the matrices B_i $(i = 1, 2, 3)$ instead of T when the nth row of B_i is given by

$$x_{kn} = \cos\left(\frac{2k-1}{2n-1}\pi\right) \qquad (k = 1, ..., n) \text{ for } i = 1, \tag{1.9}$$

$$x_{kn} = \cos\left(\frac{2k}{2n-1}\pi\right) \qquad (k = 1, ..., n) \text{ for } i = 2, \tag{1.10}$$

$$x_{kn} = \cos\left(\frac{k-1}{n-1}\pi\right) \qquad (k = 1, ..., n) \text{ for } i = 3. \tag{1.11}$$

1.4. In this paper we show that (1.8) is not true for arbitrary $f \in C(K)$ whenever the exceptional-point sequence $(x_{j(n)})_{n\in\mathbb{N}}$ satisfies (1.7).

Regarding ε-restricted sequences $(x_{j(n)})_{n\in\mathbb{N}}$ we will prove that for every $f\in V_i$ (where V_i $(i=1,2,3)$ are suitably chosen subspaces of $C(K)$) integral conditions similar to (1.6) are necessary and sufficient in order that (1.8) holds.

2. Results

Definition 2.1. For arbitrary real numbers $p, q \geqslant 0$ we define subspaces $U^{(p,q)}(K)$ of $C(K)$ by

$$f\in U^{(p,q)}(K)$$
$$\underset{\text{def}}{\Leftrightarrow} f\in C(K)$$

and

if $p>0$ there exist real numbers $M,\delta>0$ such that

$$|f(x)-f(1)|\leqslant M\cdot(1-x)^{p+\delta}$$

for every $x\in K$; and

if $q>0$ there exist real numbers $M,\delta>0$ such that

$$|f(x)-f(-1)|\leqslant M\cdot(1+x)^{q+\delta}$$

for every $x\in K$.

Theorem 2.2. *Define for abbreviation*

$$V_1 := U^{(0,1/2)}(K),\ V_2 := U^{(1/2,0)}(K),\ V_3 := U^{(1/2,1/2)}(K)$$

and

$$I(f,B_1) := \int_{-1}^{1}\frac{f(x)-f(-1)}{\sqrt{1-x^2}\cdot(1+x)}\,dx \qquad \textit{for}\quad f\in V_1,$$

$$I(f,B_2) := \int_{-1}^{1}\frac{f(x)-f(1)}{\sqrt{1-x^2}\cdot(1-x)}\,dx \qquad \textit{for}\quad f\in V_2,$$

and

$$I(f,B_3) := I(f,B_1)-I(f,B_2) \qquad \textit{for}\quad f\in V_3.$$

Consider exceptional-point sequences satisfying (1.7). *Then for every* $i=1,2,3$ *and every* $f\in V_i$ *the equivalence*

$$\lim_{n\to\infty}\|H^*_{j(n)}(f,B_i)-f\|=0 \Leftrightarrow I(f,B_i)=0$$

holds.

3. Proof of the Theorem

3.1. It is well known that for every $i=1,2,3$ and every $f \in C(K)$,

$$\lim_{n \to \infty} \|H_n(f, B_i) - f\| = 0 \tag{3.1}$$

(see V. Kumar [1] for $i=1,2$ and R. B. Saxena [3] for $i=3$). Thus, by (1.3) and (3.1), we have to verify

$$\lim_{n \to \infty} \|R_{j(n)}(f, B_i)\| = 0 \Leftrightarrow I(f, B_i) = 0 \tag{3.2}$$

for every $i=1,2,3$ and every $f \in V_i$ in order to prove the theorem. In the sequel we will verify (3.2) for $i=1$. The proofs for $i=2,3$ follow similarly.

3.2. Let $a, \beta > -1$ denote $w_{a,\beta}: K \to \mathbb{R}$ the weight function given by

$$w_{a,\beta}(x) = (1-x)^a \cdot (1+x)^\beta$$

and $P_n^{(a,\beta)}$ the nth Jacobi polynomial satisfying

$$P_n^{(a,\beta)}(1) = \binom{n+a}{n}, \tag{3.3}$$

$$P_n^{(a,\beta)}(x) = (-1)^n \cdot P_n^{(\beta,a)}(-x) \qquad (x \in K), \tag{3.4}$$

$$P_n^{(a,\beta)\prime}(x) = \tfrac{1}{2} \cdot (n+a+\beta+1) \cdot P_{n-1}^{(a+1,\beta+1)}(x) \qquad (x \in K), \tag{3.5}$$

and

$$|P_n^{(a,\beta)}(\cos \Theta)| = \begin{cases} \mathcal{O}(n^a) & \left(0 \leqslant \Theta \leqslant \dfrac{c}{n}\right), \\ \Theta^{-a-1/2} \cdot \mathcal{O}(n^{-1/2}) & \left(\dfrac{c}{n} \leqslant \Theta \leqslant \dfrac{\pi}{2}\right), \end{cases} \tag{3.6}$$

for $n \to \infty$, where $c > 0$ is arbitrarily fixed (compare G. Szegö [4, Theorem 7.32.2]).

3.3. Using (1.4) and the fact that (for $n \geqslant 2$) the nodes $x_k = \cos((2k-1)\pi/(2n-1))$ $(k=1, ..., n-1)$ are the zeros of $P_{n-1}^{(-1/2,1/2)}$ we have

$$R_{j(n)}(f, B_1) = F_{j(n)}(B_1) \cdot S_n(f, B_1), \tag{3.7}$$

where

$$F_{j(n)}(B_1, x) = \frac{w_{0,2}(x) \cdot (P_{n-1}^{(-1/2,1/2)}(x))^2}{x - x_{j(n)}} \qquad (x \in K) \tag{3.8}$$

and

$$S_n(f, B_1) = \sum_{k=1}^{n} \frac{f(x_k)}{w_{1,3}(x_k) \cdot (P_{n-1}^{(-1/2,1/2)}(x_k))^2} + \frac{2 \cdot P_{n-1}^{(-1/2,1/2)\prime}(-1)}{(P_{n-1}^{(-1/2,1/2)}(-1))^3} \cdot f(-1). \tag{3.9}$$

3.4. In order to prove (3.2) for the point system B_1 we establish the following lemmas:

LEMMA 3.1. *For every exceptional-point sequence* $(x_{j(n)})_{n \in \mathbb{N}}$ *satisfying* (1.7) *the estimate*

$$\|F_{j(n)}(B_1)\| = \mathcal{O}(1)$$

holds for $n \to \infty$.

Proof. Let $(x_{j(n)})_{n \in \mathbb{N}}$ (satisfying (1.7)) be given and consider arbitray $x \in K$.

1*st case.* $|x - x_{j(n)}| \leqslant \varepsilon/2$ (where $0 < \varepsilon < 1$ is fixed by (1.7)). By the mean value theorem of differential calculus and (3.5)

$$\begin{aligned} |F_{j(n)}(B_1, x)| &= w_{0,2}(x) \cdot |P_{n-1}^{(-1/2,1/2)}(x)| \\ &\quad \cdot \left| \frac{P_{n-1}^{(-1/2,1/2)}(x) - P_{n-1}^{(-1/2,1/2)}(x_{j(n)})}{x - x_{j(n)}} \right| \\ &= w_{0,2}(x) \cdot |P_{n-1}^{(-1/2,1/2)}(x)| \cdot |P_{n-1}^{(-1/2,1/2)\prime}(\xi)| \\ &= w_{0,2}(x) \cdot |P_{n-1}^{(-1/2,1/2)}(x)| \cdot \frac{n}{2} \cdot |P_{n-2}^{(1/2,3/2)}(\xi)| \end{aligned}$$

holds, where $\xi = \xi_{x, x_{j(n)}}$. Using the fact that $x, \xi \in [-1 + \varepsilon/2, 1 - \varepsilon/2]$ for all sufficiently large n we obtain by (3.6) and (3.4) for $n \to \infty$

$$|F_{j(n)}(B_1, x)| = \mathcal{O}(1).$$

(The constant is independent of x.)

2*nd case.* $|x - x_{j(n)}| > \varepsilon/2$. By (3.6) and (3.4) we have

$$|P_n^{(-1/2,1/2)}(\cos \Theta)|^2 = \begin{cases} \mathcal{O}(n^{-1}) & \left(0 \leqslant \Theta \leqslant \frac{\pi}{2}\right), \\ (\pi - \Theta)^{-2} \cdot \mathcal{O}(n^{-1}) & \left(\frac{\pi}{2} \leqslant \Theta \leqslant \pi - \frac{c}{n}\right), \\ \mathcal{O}(n) & \left(\pi - \frac{c}{n} \leqslant \Theta \leqslant \pi\right), \end{cases} \tag{3.10}$$

for $n \to \infty$, where $c > 0$ is arbitrarily fixed. Using the fact that

$$(1+\cos\Theta)^2 = 4\cdot\cos^4\left(\frac{\Theta}{2}\right) \leqslant \frac{1}{4}(\pi-\Theta)^4 \qquad (0 \leqslant \Theta \leqslant \pi),$$

we obtain by (3.10) that

$$\|w_{0,2}\cdot(P_{n-1}^{(-1/2,1/2)})^2\| = \mathcal{O}(n^{-1})$$

holds for $n \to \infty$. Thus we have for $|x - x_{j(n)}| > \varepsilon/2$

$$|F_{j(n)}(B_1, x)| < \frac{2}{\varepsilon}\cdot w_{0,2}(x)\cdot(P_{n-1}^{(-1/2,1/2)}(x))^2 = \mathcal{O}(n^{-1})$$

for $n \to \infty$, which completes the proof of Lemma 3.1. ∎

In order to exclude that $\lim_{n\to\infty} \|F_{j(n)}(B_1)\| = 0$ is possible we prove

LEMMA 3.2. *For every exceptional-point sequence $(x_{j(n)})_{n\in\mathbb{N}}$ satisfying (1.7) there exists a sequence $(n_k)_{k\in\mathbb{N}} \subset \mathbb{N}$, a sequence $(t_{n_k})_{k\in\mathbb{N}}$ of real numbers $t_{n_k} \in [0, \pi]$, and a fixed $C > 0$ such that*

$$|F_{j(n_k)}(B_1, \cos(t_{n_k}))| \geqslant C \tag{3.11}$$

holds for all sufficiently large n_k.

Proof. Using the asymptotic formula of Darboux for the Jacobi polynomials (compare G. Szegö [4, Theorem 8.21.8]) we obtain for $\delta \leqslant \Theta \leqslant \pi - \delta$ ($0 < \delta < \pi/2$ is arbitrarily fixed) uniformly for $n \to \infty$

$$|F_{j(n)}(B_1, \cos\Theta)| = \left|\frac{4\cdot\cos^2(\Theta/2)\cdot\cos^2(N\Theta) + \mathcal{O}(n^{-1})}{\pi\cdot(n-1)\cdot(\cos(\Theta) - \cos(\Theta_{j(n)}))}\right|, \tag{3.12}$$

where $\Theta_{j(n)} = \arccos(x_{j(n)})$ and $N = (2n-1)/2$. This asymptotic representation will be fundamental to prove Lemma 3.2. Let $(x_{j(n)})_{n\in\mathbb{N}}$ (satisfying (1.7)) be arbitrarily chosen and $y_{j(n)} := \cos(N\Theta_{j(n)})$. Then the theorem of Bolzano–Weierstraß yields

$$\lim_{k\to\infty} y_{j(n_k)} = g \in K \qquad \text{for suitable } (n_k)_{k\in\mathbb{N}} \subset \mathbb{N}.$$

Now we will construct the sequence $(t_{n_k})_{k\in\mathbb{N}}$. For that purpose we consider two cases and omit the index k of the numbers n_k in the sequel of this proof.

1st case. $g \neq 0$. We define $t_n := \Theta_{j(n)} + \pi/(n-1)$. Using

$$\cos^2(Nt_n) = \cos^2\left(N\Theta_{j(n)} + \pi + \frac{\pi}{2n-2}\right)$$

$$= \left(\cos(N\Theta_{j(n)}) \cdot \cos\left(\pi + \frac{\pi}{2n-2}\right) - \sin(N\Theta_{j(n)}) \cdot \sin\left(\pi + \frac{\pi}{2n-2}\right)\right)^2$$

we obtain

$$\lim_{n \to \infty} \cos^2(Nt_n) = g^2 > 0. \tag{3.13}$$

2nd case. $g = 0$. We define $t_n := \Theta_{j(n)} + \pi/(2n-2)$. Now

$$\cos^2(Nt_n) = \left(\cos(N\Theta_{j(n)}) \cdot \cos\left(\frac{\pi}{2} + \frac{\pi}{4n-4}\right) - \sin(N\Theta_{j(n)}) \cdot \sin\left(\frac{\pi}{2} + \frac{\pi}{4n-4}\right)\right)^2$$

and $\lim_{n \to \infty} \sin^2(N\Theta_{j(n)}) = \lim_{n \to \infty}(1 - \cos^2(N\Theta_{j(n)})) = 1$ yield

$$\lim_{n \to \infty} \cos^2(Nt_n) = 1. \tag{3.14}$$

Regarding (1.7) the existence of $0 < \delta = \delta(\varepsilon) < \pi/2$ is ensured in order that $\delta \leqslant t_n \leqslant \pi - \delta$ for both cases and all sufficiently large n hold. Thus we obtain (applying (3.12) and by (3.13), (3.14)) that

$$|F_{j(n)}(B_1, \cos(t_n))| \geqslant \left| \frac{C_1}{\pi \cdot (n-1) \cdot (\cos(t_n) - \cos(\Theta_{j(n)}))} \right| \tag{3.15}$$

is true for a suitable $C_1 > 0$ (independent of n) and all sufficiently large n. By the mean value theorem of differential calculus we obtain for those n

$$|\pi \cdot (n-1) \cdot (\cos(t_n) - \cos(\Theta_{j(n)}))| = C_2 \cdot |\sin(\xi)| \qquad (\delta < \xi = \xi(t_n, \Theta_{j(n)}) < \pi - \delta) \leqslant C_2. \tag{3.16}$$

(For the 1st case we have $C_2 = \pi^2$ and for the 2nd case $C_2 = \pi^2/2$.) The estimates (3.15) and (3.16) complete the proof of Lemma 3.2. ∎

LEMMA 3.3. *For every $f \in V_1$ the representation*

$$S_n(f, B_1) = \sum_{k=1}^{n} \frac{f(x_k) - f(-1)}{w_{1,3}(x_k) \cdot (P_{n-1}^{(-1/2,1/2)\prime}(x_k))^2} \tag{3.17}$$

holds.

Proof. We will prove

$$\frac{2 \cdot P_{n-1}^{(-1/2,1/2)\prime}(-1)}{(P_{n-1}^{(-1/2,1/2)}(-1))^3} = -\sum_{k=1}^{n} \frac{1}{w_{1,3}(x_k) \cdot (P_{n-1}^{(-1/2,1/2)\prime}(x_k))^2}, \tag{3.18}$$

then (3.17) follows immediately by (3.9). For the polynomial

$$G_{n-1}(x) = \frac{\cos(2n-1)/2 \cdot \arccos(x))}{\cos(\arccos(x)/2)} \qquad (x \in K),$$

we have

$$G'_{n-1}(x_k) = \frac{2n-1}{2} \cdot \frac{\sin((2n-1)/2 \cdot \arccos(x_k))}{\sqrt{1-x_k^2} \cdot \cos(\arccos(x_k)/2)} \quad (k = 1, ..., n-1)$$

and

$$(G'_{n-1}(x_k))^2 = \frac{(2n-1)^2}{2 \cdot w_{1,2}(x_k)} \qquad (k = 1, ..., n-1). \tag{3.19}$$

On the other hand we have

$$(G_{n-1}(-1))^2 = (2n-1)^2. \tag{3.20}$$

Using the fact that

$$P_{n-1}^{(-1/2,1/2)}(x) = \prod_{k=1}^{n-1} ((2k-1)/2k) \cdot G_{n-1}(x),$$

by (3.19) and (3.20) it follows that

$$(P_{n-1}^{(-1/2,1/2)}(-1))^2 = 2 \cdot w_{1,2}(x_k) \cdot (P_{n-1}^{(-1/2,1/2)\prime}(x_k))^2 \tag{3.21}$$

is true for every $k = 1, ..., n-1$. Now by (3.21) and

$$\frac{P_{n-1}^{(-1/2,1/2)\prime}(-1)}{P_{n-1}^{(-1/2,1/2)}(-1)} = -\sum_{k=1}^{n-1} \frac{1}{1+x_k}$$

(3.18) follows immediately. ∎

LEMMA 3.4. *For every $f \in V_1$ we have*

$$\lim_{n \to \infty} S_n(f, B_1) = \tfrac{1}{2} \cdot I(f, B_1).$$

Proof. Consider for $\alpha, \beta > -1$ the nth Gauss–Jacobi quadrature formula (see G. Szegö [4, Ch. 15.3])

$$Q_n^{(\alpha,\beta)}(g) = \sum_{k=1}^{n} \lambda_{kn}^{(\alpha,\beta)} \cdot g(x_{kn}^{(\alpha,\beta)}) \tag{3.22}$$

in respect to $w_{\alpha,\beta}$, where $g: K \to \mathbb{R}$ is a function for which the (possibly improper) integral

$$I^{(\alpha,\beta)}(g) := \int_{-1}^{1} w_{\alpha,\beta}(x) \cdot g(x)\, dx$$

exists.

The weights $\lambda_{kn}^{(\alpha,\beta)}$ $(k = 1, ..., n)$ of (3.22) are given by

$$\lambda_{kn}^{(\alpha,\beta)} = \frac{\gamma_n^{(\alpha,\beta)}}{w_{1,1}(x_{kn}^{(\alpha,\beta)}) \cdot (P_n^{(\alpha,\beta)\prime}(x_{kn}^{(\alpha,\beta)}))^2}, \tag{3.23}$$

where

$$\gamma_n^{(\alpha,\beta)} = 2^{\alpha+\beta+1} \cdot \frac{\Gamma(n+\alpha+1) \cdot \Gamma(n+\beta+1)}{\Gamma(n+1) \cdot \Gamma(n+\alpha+\beta+1)} \tag{3.24}$$

and $x_{kn}^{(\alpha,\beta)}$ $(k = 1, ..., n)$ are the zeros of $P_n^{(\alpha,\beta)}$. It is well known that the quadrature convergence

$$\lim_{n \to \infty} Q_n^{(\alpha,\beta)}(g) = I^{(\alpha,\beta)}(g) \tag{3.25}$$

holds for every function $g: K \to \mathbb{R}$, for which the integral $I^{(\alpha,\beta)}(g)$ exists (compare G. Szegö [4, Theorem 15.2.3]). Now let $f \in V_1$ be arbitrarily fixed. By (3.17), (3.22)–(3.24) we obtain

$$S_n(f, B_1) = \frac{1}{\gamma_{n-1}^{(-1/2, 1/2)}} \cdot Q_{n-1}^{(-1/2, 1/2)}(g_f), \tag{3.26}$$

where $g_f :]-1, 1] \to \mathbb{R}$ is defined by

$$g_f(x) := \frac{f(x) - f(-1)}{w_{0,2}(x)}.$$

The assumption $f \in V_1$ implies the existence of real constants M, $\delta > 0$ such that

$$\left| \frac{f(x) - f(-1)}{(1+x)^{\delta + 1/2}} \right| \leqslant M$$

is true for every $x \in]-1, 1]$, which implies

$$|w_{-1/2,1/2}(x) \cdot g_f(x)| = w_{-1/2,\delta-1}(x) \cdot \left| \frac{f(x) - f(-1)}{(1+x)^{\delta+1/2}} \right|$$
$$\leqslant w_{-1/2,\delta-1}(x) \cdot M \qquad \text{for} \quad x \in]-1, 1].$$

This estimate ensures (using the existence of the integral $\int_{-1}^{1} w_{-1/2,\delta-1}(x) \cdot M\, dx$) the existence of $I^{(-1/2,1/2)}(g_f)$. Thus (3.25) can be applied; regarding (3.26) and $\lim_{n\to\infty} \gamma_n^{(-1/2,1/2)} = 2$ we obtain

$$\lim_{n\to\infty} S_n(f, B_1) = \lim_{n\to\infty} \frac{1}{\gamma_n^{(-1/2,1/2)}} \cdot Q_{n-1}^{(-1/2,1/2)}(g_f)$$
$$= \tfrac{1}{2} \cdot I^{(-1/2,1/2)}(g_f) = \tfrac{1}{2} \cdot I(f, B_1). \quad \blacksquare$$

3.5. Now (3.2) can easily be proved by use of the preceding lemmas. Therefore let an exceptional-point sequence $(x_{j(n)})_{n \in \mathbb{N}}$ satisfying (1.7) and an $f \in V_1$ be arbitrarily given.

(i) In the case $I(f, B_1) = 0$ Lemma 3.4 implies $\lim_{n\to\infty} S_n(f, B_1) = 0$ which yields (by Lemmas 3.1 and (3.7)) $\lim_{n\to\infty} \|R_{j(n)}(f, B_1)\| = 0$.

(ii) On the other hand let be $\lim_{n\to\infty} \|R_{j(n)}(f, B_1)\| = 0$. This implies $\lim_{n\to\infty} \|F_{j(n)}(B_1)\| \cdot S_n(f, B_1) = 0$. That yields (using that $\lim_{n\to\infty} \|F_{j(n)}(B_1)\| = 0$ is impossible by Lemma 3.2)

$$\lim_{n\to\infty} S_n(f, B_1) = \tfrac{1}{2} \cdot I(f, B_1) = 0. \quad \blacksquare$$

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